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Hirota method for oblique solitons in two-dimensional supersonic nonlinear Schrödinger flow

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ABSTRACT

In a previous work El et al. (2006) [1] exact stable oblique soliton solutions were revealed in two-dimensional nonlinear Schrödinger flow. In this work we show that single soliton solution can be expressed within the Hirota bilinear formalism. An attempt to build two-soliton solutions shows that the system is "close" to integrability provided that the angle between the solitons is small and/or we are in the hypersonic limit.

1. Introduction

The nonlinear Schrödinger (NLS) flow is ubiquitous in many physical systems such as photorefractive crystals, and the superfluids Bose–Einstein condensates and exciton-polaritons. A fundamental problem is how a superfluid reacts to the presence of an obstacle. One can define a Mach velocity (M) as the velocity of the obstacle relative to the sound velocity in the medium. Two-dimensional studies showed that when $M > 0.37$ the system loses superfluidity and start to emit pair of vortices [2–5]. Increasing the velocity showed that vortices merge in a "vortex street" [6], which were later understood as oblique solitons [7] and its exact single soliton solution determined [1,8]. Oblique solitons were long know to be unstable but it was showed that under the flow they are only convectively unstable provided that $M > 1.44$ [9,10]. Studies with extended obstacles also presented oblique solitons in the wake, and an analytical approach based on Whitham modulation theory was successfully applied [11]. Oblique solitons were realized experimentally in the system of exciton-polaritons [12], though for lower Mach number than originally predicted by theory. Corrections to the model including losses were able to match experimental observations [13]. Dynamics of formation and decay of oblique solitons were recently observed in [14].

A key question about solitons is how they behave in collisions. As long as we know, there is no general proof about the non-integrability of the 2D-NLS. Numerical studies with two obstacles were able to generate collisions between these oblique solitons [15]. These collisions were shown to be practically elastic suggesting integrability or "close" to integrability in such system. In the same work, an analytical treatment was considered using hydrodynamical approach and the system was show to follow a 1D-NLS equation in the hypersonic limit, and collisions could be described by the well known phase shifts [16]. Numerical calculations were in good agreement with predicted phase shifts, considering that they were perturbed by previous interactions with linear waves. Since the exact single soliton (1SS) was already obtained, one might ask if an exact two-soliton solution (2SS) could be found. A possible framework to find multiple soliton solutions is the Hirota method [17,18]. In the following we build up a bilinear Hirota form of the 2D-NLS in the stationary frame relative to the obstacle. Then, we show that the single oblique soliton solution indeed satisfy this form. In the sequence we propose an ansatz to the exact solution of the two-soliton interaction problem and analyze its consequences within this formalism.

2. Model

Oblique dark solitons in a superfluid are described [1] as stationary solutions of the defocusing nonlinear Schrödinger equation (NLS)

$$i\psi_t = -\frac{1}{2}\Delta\psi + |\psi|^2\psi + V(x+Mt, y)\psi, \tag{1}$$

which is written here in standard dimensionless units, $\Delta \equiv \partial_x^2 + \partial_y^2$, the subscripts mean derivatives and the potential V is modeled as a small impenetrable obstacle. The potential V is moved with Mach velocity M from right to left across the fluid, where M is in units of the sound velocity. We make a global phase transformation $\psi' = e^{it}\psi$ and later a Galilean transformation $x' = x + Mt$, $t' = t$ leading to

$$-2i\psi_t = \psi_{xx} + \psi_{yy} + 2iM\psi_x + 2\psi - 2\psi|\psi|^2 - 2V(x, y)\psi, \tag{2}$$

where the primes were omitted for convenience.

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This last equation describes the wave function in the stationary frame relative to the obstacle. Also the boundary condition is $\psi = 1$ as $x, y \to \pm\infty$. We assume that for time long enough the system relaxes to a stationary solution, i.e., $\psi_t = 0$ is satisfied. This is well verified for supersonic flow ($M > 1.44$) and in the following it will be enough to find oblique soliton solutions. Using this condition, we express the wave function as $\psi = G/F$ and substitute in Eq. (2). Multiplying the resulting equation by F^3 one obtains

$$\begin{aligned} &F\big[G_{xx}F - 2G_xF_x - GF_{xx} + 2iM(G_xF - GF_x) + 2GF + G_{yy}F \\ &\quad - 2G_yF_y - GF_{yy}\big] + 2GF_xF_x + 2GF_yF_y - 2|G|^2G = 0, \end{aligned} \tag{3}$$

where the potential V is omitted since we will look for solutions after passing the obstacle [1].

We now use the well known Hirota techniques. We make the replacements $-GF_{xx} \to +GF_{xx} - 2GF_{xx}$ and $-GF_{yy} \to +GF_{yy} - 2GF_{yy}$, multiply by F and rearrange the equation as

$$\begin{aligned} &F^2\big[\big(2iMD_x + D_x^2 + D_y^2\big)G.F + 2GF\big] \\ &\quad - GF\big[\big(D_x^2 + D_y^2\big)F.F + 2|G|^2\big] = 0, \end{aligned} \tag{4}$$

where the Hirota D-operator is defined generally as $D_x^n f.g \equiv (\partial_x - \partial_{x'})^n g(x) f(x')|_{x=x'}$. In our particular case $D_xG.F \equiv G_xF - GF_x$, $D_x^2G.F = G_{xx}F - 2G_xF_x + GF_{xx}$, $D_x^2F.F = 2FF_{xx} - 2F_xF_x$.

Eq. (4) suggests that the system can be put in bilinear form as

$$\big(2iMD_x + D_x^2 + D_y^2\big)G.F + 2GF = \Lambda GF, \tag{5}$$

$$\big(D_x^2 + D_y^2\big)F.F + 2GG^* = \Lambda F^2, \tag{6}$$

where Λ is a constant to be determined. This system of equations have very close similarity to the bilinear form of dark solitons in 1D-NLS [17,18].

3. Single oblique soliton solution

The single oblique soliton solution was already found in [1,15] assuming null vorticity and using a hydrodynamic formalism. One can write it in the stationary frame as

$$\psi = \frac{\nu(e^{\xi/2} - e^{-\xi/2}) - i\lambda(e^{\xi/2} + e^{-\xi/2})}{e^{\xi/2} + e^{-\xi/2}}, \tag{7}$$

where $\xi \equiv 2\nu[x\sin\theta - y\cos\theta]$, $\nu \equiv \sqrt{1-\lambda^2}$, $\lambda \equiv M\sin\theta$, and θ is the angle between the soliton and the horizontal axis. $M\sin\theta = \pm 1$ defines the Mach cone and thus solitons can be found only in the region $-\arcsin(1/M) < \theta < \arcsin(1/M)$.

Multiplying Eq. (7) by an ineffective global phase $i(\lambda + i\nu)$ and numerator and denominator by $e^{\xi/2}$ we have

$$\psi = \frac{1 + e^{\xi + 2i\alpha}}{1 + e^{\xi}}, \tag{8}$$

where $e^{i\alpha} \equiv \lambda + i\nu$. One can now readily identify the functions

$$G = 1 + e^{\xi + 2i\alpha}, \tag{9}$$

$$F = 1 + e^{\xi}. \tag{10}$$

After substitution of the functions G and F in Eqs. (5), (6) one finds that they remarkably satisfy the bilinear equations provided that $\Lambda = 2$. Thus, we were able to show that the single soliton solution can be built within the Hirota method. It is now natural to look for multiple soliton solutions using this formalism. We will pursue this in the following.

4. Ansatz for two-soliton solution

Based on the similarity of Eqs. (5), (6) with the 1D-NLS bilinear form and dark soliton solution [19], we build an ansatz for the two-soliton solution in 2D-NLS supersonic flow as

$$G = 1 + e^{\xi_1 + 2i\alpha_1} + e^{\xi_2 + 2i\alpha_2} + e^{\xi_1 + \xi_2 + 2i\alpha_1 + 2i\alpha_2 + \varphi_{12}}, \tag{11}$$

$$F = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + \varphi_{12}}, \tag{12}$$

where $\xi_j = 2\nu[x\sin\theta_j - y\cos\theta_j]$, $\nu_j = \sqrt{1-\lambda_j^2}$, $\exp(i\alpha_j) = \lambda_j + i\nu_j$, $j = 1, 2$ and φ_{12} is to be determined.

Then, we substitute the ansatz (11), (12) in Eqs. (5) and (6) and collect terms proportional to e^0, e^{ξ_j}, $e^{2\xi_j}$, $e^{\xi_j+\xi_k}$, $e^{2\xi_j+\xi_k}$, $e^{2\xi_j+2\xi_k}$, $j, k = \{1, 2\}$, $j \neq k$. Terms proportional to e^0, e^{ξ_j}, $e^{2\xi_j}$ well satisfy the bilinear equations since they correspond to single soliton solutions. The term $e^{2\xi_1+\xi_2}$ gives equations

$$e^{2i\alpha_1+\varphi_{12}}\big[4i\nu_2\lambda_2\big(e^{2i\alpha_2} - 1\big) + 4\nu_2^2\big(e^{2i\alpha_2} + 1\big)\big] = 0, \tag{13}$$

$$8\nu_2^2 e^{\varphi_{12}} + 2e^{\varphi_{12}}\big(e^{-2i\alpha_2} + e^{2i\alpha_2}\big) - 4e^{\varphi_{12}} = 0, \tag{14}$$

which can be shown to be true with little algebra, independently of the value of $e^{\varphi_{12}}$. The same for the term $e^{\xi_1+2\xi_2}$. Terms proportional to $e^{2\xi_1+2\xi_2}$ give equations that are easily shown to be satisfied.

The remaining terms proportional to $e^{\xi_1+\xi_2}$ generate equations

$$\begin{aligned} &4i\big[(\nu_1\lambda_1 - \nu_2\lambda_2)e^{2i\alpha_1} + (\nu_2\lambda_2 - \nu_1\lambda_1)e^{2i\alpha_2} \\ &\quad + (\nu_1\lambda_1 + \nu_2\lambda_2)\big(e^{2i\alpha_1+2i\alpha_2} - 1\big)e^{\varphi_{12}}\big] \\ &\quad + 4\big[S_-\big(e^{2i\alpha_1} + e^{2i\alpha_2}\big) + S_+\big(e^{2i\alpha_1+2i\alpha_2+\varphi_{12}}\big) + S_+e^{\varphi_{12}}\big] = 0 \end{aligned} \tag{15}$$

and

$$\begin{aligned} &4\big(S_- + S_+e^{\varphi_{12}}\big) + e^{2i\alpha_1-2i\alpha_2} + e^{-2i\alpha_1+2i\alpha_2} \\ &\quad + \big(e^{2i\alpha_1+2i\alpha_2} + e^{-2i\alpha_1-2i\alpha_2}\big)e^{\varphi_{12}} - 2 - 2e^{\varphi_{12}} = 0, \end{aligned} \tag{16}$$

where $S_\pm \equiv \nu_1^2 \pm 2\nu_1\nu_2\sigma + \nu_2^2$, $\sigma \equiv \sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 = \cos(\theta_1 - \theta_2)$. Apart from the extra factor σ, these equations are equal to the ones extracted from the 1D-NLS [19].

Multiplying the whole equation (15) by $e^{-i\alpha_1-i\alpha_2}$ and after algebraic manipulation we get

$$e_a^{\varphi_{12}} = \frac{\sigma^{-1} - \lambda_1\lambda_2 - \nu_1\nu_2}{\sigma^{-1} - \lambda_1\lambda_2 + \nu_1\nu_2} \tag{17}$$

and Eq. (16) gives

$$e_b^{\varphi_{12}} = \frac{\sigma - \lambda_1\lambda_2 - \nu_1\nu_2}{\sigma - \lambda_1\lambda_2 + \nu_1\nu_2}, \tag{18}$$

where the subscripts a and b correspond to $e^{\varphi_{12}}$ extracted from Eqs. (15) and (16), respectively.

For 1D-NLS, σ is equal to 1 and thus $e_a^{\varphi_{12}} = e_b^{\varphi_{12}}$ and the two-soliton solution is integrable. If $|\theta_1 - \theta_2|$ is sufficiently small then $\sigma \sim 1$ and the collision of two solitons must be practically elastic. Here the treatment is developed regardless the value of M. In the symmetrical case, i.e., $\theta_1 = -\theta_2 = \theta$, one can examine the ratio $R = e_b^{\varphi_{12}}/e_a^{\varphi_{12}}$, this should provide a measure of how close to the Hirota integrability is the proposed ansatz. In Fig. 1 we show this ratio as a function of $\sin\theta$ for different flow velocities M. For $M = 2$ and small θ the system is far from integrability and the ansatz is poor especially at the collision region and get strong deviations from phase shifts. As M is increased the ratio R is closer to 1, independent of the angle θ. This was already anticipated in Ref. [15] using a hydrodynamical approach, where it was assumed

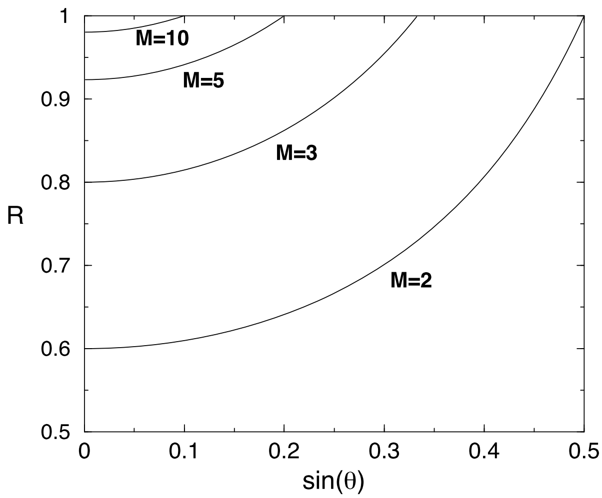

Fig. 1. Results of the proposed ansatz similar to the 1D-NLS two dark soliton solution. The ratio $R = e_b^{\varphi_{12}}/e_a^{\varphi_{12}}$ is between the factors that satisfy each one of the two Hirota bilinear equations.

hypersonic limit ($M \gg 1$) so that the equation can be approximated to 1D-NLS. For the typical case of an impenetrable obstacle with radius $r = 1$, solitons are generated with $\sin\theta \sim 0.1$ [1], that gives $R \sim 0.93$ and the collision shall appear as almost elastic. This is consistent with the observations in Ref. [15].

5. Phase shifts

A typical behavior of solitons is that they maintain their shapes after collisions. However, their positions just after collision are dislocated relative to the free soliton propagation. These dislocations are named *phase shifts* and are closely related to the factor $\exp(\varphi_{12})$. One can calculate the phase shift keeping one of the solitons fixed and observing its position when the other soliton is located at infinity [20]. Following this recipe, we take the ansatz of $\psi = G/F$ given by Eqs. (11), (12), multiply numerator and denominator by $e^{-\xi_2}$ and take the limit $\xi_2 \to \infty$ giving

$$\psi \sim e^{2i\alpha_2} \frac{(1 + e^{\xi_1 + 2i\alpha_1 + \varphi_{12}})}{1 + e^{\xi_1 + \varphi_{12}}}. \tag{19}$$

Thus, comparing with the single soliton solution from Eq. (7), the first soliton that depends on ξ_1 suffers translation as $\xi_1 \to \xi_1 + \varphi_{12}$. The dislocation δy can be calculated simply by

$$2\nu_1[x\sin\theta_1 - y\cos\theta_1] + \varphi_{12} = 2\nu_1\big[x\sin\theta_1 - (y + \delta y)\cos\theta_1\big] \tag{20}$$

giving

$$\delta y = \frac{-\varphi_{12}}{2\nu_1 \cos\theta_1}. \tag{21}$$

Finally, using expression (17) in the limit $\sigma \sim 1$ gives

$$\delta y = \frac{-1}{2\nu_1 \cos\theta_1} \ln\left[\frac{1 - \lambda_1\lambda_2 - \nu_1\nu_2}{1 - \lambda_1\lambda_2 + \nu_1\nu_2}\right]. \tag{22}$$

In the hypersonic limit with $M \gg 1$, $\cos\theta_1$ tends to one, and we recover the well known formula of phase shift of 1D dark solitons [15,16]. Analogous results are obtained for the phase shift of the second soliton.

6. Conclusions

We studied the problem of oblique solitons solutions in two-dimensional NLS flow using the Hirota bilinear form. This derivation was made in the obstacle frame and assuming stationary flow. We were able to build exact single soliton solution with similar form of the 1D-NLS. An ansatz for the two-soliton solution was proposed. It is shown that for high Mach number the collision can be considered as practically elastic and amplitude and phase can be predicted from NLS-1D approximation, in agreement with previous hydrodynamical approach and numerical simulations. Also, solitons with small angles between them will collide almost elastically, regardless the velocity M. These results are relevant for possible experiments like generation of oblique solitons with exciton-polaritons that were recently reported in [12,14].

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