Stable multiple vortices in collisionally inhomogeneous attractive Bose-Einstein condensates

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We study the stability of solitary vortices in a two-dimensional trapped Bose-Einstein condensate (BEC) with a spatially localized region of self-attraction. Solving the respective Bogoliubov–de Gennes equations and running direct simulations of the underlying Gross-Pitaevskii equation reveals that vortices with a topological charge up to $S = 6$ (at least) are stable above a critical value of the chemical potential (i.e., below a critical number of atoms, which sharply increases with S). The largest nonlinearity-localization radius admitting stabilization of higher-order vortices is estimated analytically and accurately identified in numerical form. To the best of our knowledge, this is the first example of a setting which gives rise to *stable* higher-order vortices, $S > 1$, in a trapped self-attractive BEC. The same setting may be realized in nonlinear optics too.

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Introduction. The creation of atomic Bose-Einstein condensates (BECs) [1], and subsequent identification of nonlinear excitations in them, such as bright [2,3] and dark [4,5] solitons in quasi-one-dimensional (1D) "cigar-shaped" settings, has triggered a great deal of interest in the investigation of matter-wave dynamics in the BEC, which is very accurately modeled by the Gross-Pitaevskii equation (GPE) [6–9]. In its general form, it is tantamount to the 3D nonlinear Schrödinger equation, which includes a trapping potential and a cubic term accounting for the mean-field nonlinearity.

The discovery of robust coherent nonlinear excitations, in the form of *dromions* [10] (localized 2D patterns produced by overlapping of 1D ghost solitons) and *lumps* (weakly localized 2D solitons) [11] in other 2D models has prompted a search for similar nonlinear excitations in BEC as well, following the investigation of more straightforward nonlinear modes—in particular, vortices [12] and Faraday waves [14]—in effectively 2D "pancake-shaped" BECs. Faraday waves are undulating nonlinear excitations generated by time-periodic shaking of the trapping potential, while vortices are created by stirring the condensate with the help of properly designed laser beams, by coherent transfer of orbital angular momentum to the condensate by the two-photon stimulated Raman process [12,13], or by imprinting of an appropriate phase pattern onto a trapped condensate [15] (see the recent survey in Ref. [16]). While in self-repulsive condensates, simple vortices (with topological charge 1) are normally stable, their stability in attractive BECs is a challenging problem, as the self-attraction gives rise to both collapse and azimuthal instability of solitary vortices, thus easily destroying them [17–26]. Vortex solitons tend to be unstable even in media where the collapse does not occur, such as those featuring quadratic nonlinearity [27].

The identification and experimental demonstration of more complex vortical structures, such as vortex dipoles [28,29] and quadrupoles [30,31], were confined to BECs with self-repulsive nonlinearity. The most natural setting for hosting vortices is provided by a pancake-shaped axially symmetric BEC, which is strongly confined in one direction (z) and weakly confined in the transverse plane. Although stable vortices with topological charge $S = 1$ have been predicted in 2D self-attractive BECs with an in-plane trapping potential [20–22], all multiple vortices with $S \geqslant 2$ were found to be unstable in the same model. This occurs by virtue of the fact that multiple vortices are vulnerable to instability against splitting into unitary ones (in repulsive media too [13,18,19]).

Stabilization of multiple vortices is a fundamentally interesting problem [23]. The present work aims to predict stable vortices with $S \geqslant 2$ in a condensate with *spatially localized* attractive nonlinearity, which, without the help of the trapping potential, may support stable fundamental solitons (with $S = 0$) but not vortices [26,32]. To the best of our knowledge, the setting elaborated in the present work is the first system which gives rise to stable higher-order vortices in a trapped self-attractive BEC. The model may also find its realization in nonlinear optics.

The model and basic results. In the mean-field approximation, the self-attractive BEC, weakly trapped in the radial direction and strongly confined along the z axis, is governed by the 3D GPE for the single-atom wave function [6,7],

$$i\hbar\frac{\partial\Psi}{\partial T} = \left(-\frac{\hbar^2}{2m}\nabla^2_{3\mathrm{D}} + \frac{m}{2}\left[\Omega_r^2(X^2+Y^2)+\Omega_z^2 Z^2\right] + \frac{4\pi\hbar^2 a_s\mathcal{N}}{m}|\Psi|^2\right)\Psi, \tag{1}$$

where $\nabla^2_{3\mathrm{D}}$ acts on coordinates X, Y, and Z, m is the atomic mass, Ω_r and Ω_z are the frequencies of the radial and axial confinement (so that Ω_r is defined as the relative frequency), a_s is the s-wave scattering length, which is negative for attractive interatomic interactions, and $\mathcal{N}$ is the number of atoms in the condensate, the norm of the wave function being 1. Factorizing the wave function by means of the usual substitution, $\Psi(X,Y,Z,T) = \pi^{-1/4}a_z^{-3/2}\exp(-i\Omega_z t/2 - Z^2/2a_z^2)\psi(x,y,t)$, where $a_z = \sqrt{\hbar/m\Omega_z}$ is the transverse-confinement size, one integrates

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Eq. (1) over Z to derive the 2D form of the GPE [33–35] in terms of the scaled coordinates, $(x,y) \equiv (X,Y)/a_z$, $t \equiv \Omega_z T$: $i\partial\psi/\partial t = [-(1/2)\nabla^2_{2D} + (1/2)\Omega_r^2 r^2 + gN|\psi|^2]\psi$, where (r,θ) are the polar coordinates in the (x,y) plane, and $g \equiv 2a_s\sqrt{2\pi m\Omega_z/\hbar}$ is the effective strength of the nonlinearity. The 2D wave function is also subject to the unitary normalization condition, $\int\int|\psi(x,y,t)|^2 dxdy = 1$. We here assume that the atomic density is not too high, therefore deviation of the nonlinearity in the 2D equation from the cubic term [34] may be neglected.

It was predicted in various forms theoretically [26] and demonstrated experimentally [36] that the use of a Feshbach resonance controlled by laser illumination [37] makes it possible to engineer spatially inhomogeneous nonlinearity [26]. Accordingly, the modified 2D GPE is rewritten as

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\nabla^2_{2D} + \frac{1}{2}r^2 + g(r)N|\psi|^2\right)\psi, \qquad (2)$$

with the nonlinearity coefficient, g, made a function of the radial coordinate and $\Omega_r = 1$ fixed by straightforward rescaling.

Stationary vortex solutions to Eq. (2) are sought for as

$$\psi(r,\theta,t) = R(r)\exp(iS\theta - i\mu t), \qquad (3)$$

where S is the integer vorticity and μ the chemical potential. Inserting Eq. (3) into Eq. (2), we obtain an equation for amplitude $R(r)$,

$$2\mu R + R'' + r^{-1}R' - (S^2 r^{-2} + r^2)R - 2g(r)NR^3 = 0, \qquad (4)$$

which can be solved numerically.

We choose a Gaussian spatial-modulation profile for the localized self-attraction,

$$g(r) = -\exp(-b^2r^2/2), \qquad (5)$$

where b^{-1} determines the radius of the nonlinearity-bearing area, and a free coefficient in front of the Gaussian was absorbed into N in Eq. (2), hence N is proportional to the number of atoms but is not identical to it, unlike $\mathcal{N}$ in Eq. (1). Thus, there remain two free parameters in Eq. (2), b and N. While N characterizes the strength of the nonlinearity, $1/b$ determines the radius of the nonlinearity-bearing region, in units of the trapping size imposed by the in-plane (HO) potential. The case of $b^2 \ll 1$ amounts to the settings studied in Refs. [20,21], which, as stated above, admit stable trapped states solely with $S = 0$ and 1. On the other hand, $b^2 \gg 1$ implies that the system is almost linear, with a small nonlinear spot located in the center. In that case, the vortices are, essentially, the same modes as their counterparts trapped in the HO potential in the framework of linear quantum mechanics, hence they are stable. Thus, a nontrivial issue is to find a minimum value, $b_{\min}$, at which the localization of the nonlinearity leads to the stabilization of vortices with $S \geqslant 2$. The results reported below demonstrate that $b_{\min} \approx 1.0$ (in particular, for $S = 4$). This finding can be understood by noting that the radial wave function of the linear HO, $R_S(r) = (\pi S!)^{-1/2} r^S \exp(-r^2/2)$, yields the average value $\langle r^2\rangle = 1 + S$. Then the stabilization condition in the framework of Eq. (2) may be naturally defined as the attenuation of the local strength of the nonlinearity by modulation profile, (5), by an order of magnitude (or more) at the location of the vortex, $r^2 = \langle r^2\rangle$ (this mechanism is different from the centrifugal stabilization effect proposed in Ref. [17], which was relevant for uniform nonlinearity). Thus, stabilization is expected at $b \geqslant \sqrt{(2/(1+S))\ln 10}$. In particular, for $S = 4$ the estimate yields $b_{\min} \approx 0.96$, in reasonable agreement with the numerical findings.

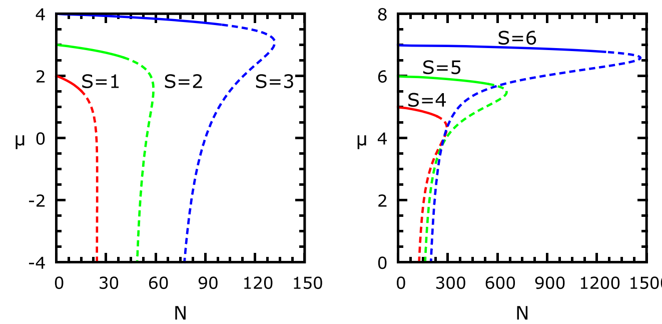

FIG. 1. (Color online) $\mu(N)$ curves for different vorticities S, as produced by the numerical solution of Eq. (4) with $b = 1.5$. Solid lines denote stable portions of the trapped-vortex families; dashed lines, unstable portions.

In addition to BECs, the model based on Eqs. (2) and (5), with t replaced by the propagation distance, may be realized in optics as a spatial-domain propagation equation in a bulk waveguide, with localized Kerr nonlinearity [26], and the linear confining potential representing transverse modulation of the refractive index in a guiding channel. Localized nonlinearity may be obtained by inhomogeneously doping the host material with elements which induce local enhancement of nonlinearity via two-photon resonance [38].

When the number of atoms is very small ($N \to 0$), the model reduces, as noted above, to the HO, with the respective eigenvalues, $\mu_{\max} = 1 + S$. The contribution of the self-focusing nonlinearity to μ is negative, therefore $\mu_{\max}$ is the upper limit for values of the chemical potential of trapped vortices.

The most essential results of the present work are summarized in Fig. 1 and Tables I and II. In the figure, families of vortices are represented (including their stability; see details below) by dint of $\mu(N)$ curves, for different values of S at $b = 1.5$. The results demonstrate that the vortices are stable at $\mu_{\min} < \mu < \mu_{\max}$, where $\mu_{\min}$ is the boundary between solid and dashed line segments in Fig. 1. At $\mu < \mu_{\min}$, the vortices are unstable, suffering splitting and collapse (details are shown below). The figure also demonstrates that values of N corresponding to $\mu_{\min}$ increase with S, which means that stable vortices with a higher topological charge may hold many more atoms than their counterparts with $S = 1$ and 2. General results for a range of relevant values of b are listed in Tables I and II.

Note that, for $S = 5$ and 6, Table II includes entries with the stability intervals starting at finite $N_{\min}$, rather than at $N = 0$. Accordingly, in Table I, these cases correspond to $\mu_{\max} < 1 + S$. This complex behavior cannot be explained by the simple arguments given above and might only be revealed by systematic numerical analysis.

These results can be translated into physical units, taking the atomic BEC of ^{7}Li as an example, with $\Omega_z/2\pi \sim 1$ kHz, which

TABLE I. Intervals ($\mu_{\min};\mu_{\max}$) in which the vortices are stable, at different values of S and b. For $S=1$, 3, and 4, only $\mu_{\min}$ is given, as $\mu_{\max}=1+S$ in these cases.

	$S=1$	$S=2$	$S=3$	$S=4$	$S=5$	$S=6$
$b=0.75$	1.21	(2.16 ; 2.27)	Unstable	Unstable	Unstable	Unstable
$b=1.0$	1.29	(2.30 ; 3)	Unstable	4.82	(5.51 ; 5.84)	(6.51 ; 6.60)
$b=1.5$	1.51	(2.58 ; 3)	3.64	4.69	(5.73 ; 6)	(6.76 ; 7)
$b=2.0$	1.66	(2.73 ; 3)	3.79	4.83	(5.85 ; 6)	(6.84 ; 6.96)

corresponds to $a_z\sim 3\ \mu$m, and the in-plane trapping frequency $\sim$10 Hz. Then the actual number of atoms is estimated as $\sim 500N$ (N are the values listed in Fig. 1 and Table II), and the size of the vortices, which are shown in Figs. 3 and 4, is obtained from the respective scaled lengths by multiplying them by $\sim 30\ \mu$m. Thus, the critical values of the atom number, corresponding to the data presented in Table II, range from $\sim$5000 for $S=1$ to $\sim 5\times 10^7$ for $S=6$. The latter estimate implies that practically all the vortices with high values of S, which can be created at relevant values of b, will be stable.

Linear-stability analysis. The stability results reported above were produced by taking perturbed solutions to Eq. (2) as Ref. [21]

$$\psi(r,t)=\left[R(r)+\varepsilon\sum_L u_L(r)e^{iL\theta-i\omega_L t}+\varepsilon\sum_L v_L^*(r)e^{-iL\theta+i\omega_L^* t}\right]e^{iS\theta-i\mu t},\qquad (6)$$

where (u,v) and ω_L are eigenmodes and eigenfrequencies corresponding to integer azimuthal index L of the perturbation with infinitesimal amplitude ε. The linearization around the stationary solution leads to a system of Bogoliubov–de Gennes equations [7],

$$\begin{pmatrix}\hat{D}_+ & g(r)R^2\\ -g(r)R^2 & -\hat{D}_-\end{pmatrix}\begin{pmatrix}u_L\\ v_L\end{pmatrix}=\omega_L\begin{pmatrix}u_L\\ v_L\end{pmatrix},\qquad (7)$$

$$\hat{D}_\pm\equiv-\frac{1}{2}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{(S\pm L)^2}{r^2}\right]+\frac{1}{2}\Omega_r^2r^2+2g(r)NR^2-\mu,$$

supplemented by the boundary conditions requiring $u(r)$ and $v(r)$ to decay as $r^{|S\pm L|}$ at $r\to 0$ and exponentially at $r\to\infty$.

Determination of the stability from Eq. (7) requires numerical diagonalization of the matrix. The vortex is unstable if, for a given μ, at least one pair of eigenvalues ω_L is complex. Results of the analysis are illustrated in Fig. 2, which shows the instability growth rates of perturbations, i.e., the largest imaginary part of ω_L, vs μ for $S=1,2,3$, and 6. We conclude that stability regions exist for all these values of S, as shown in Fig. 1 and Tables I and II.

Perturbed evolution of trapped vortices. To corroborate the predictions of the linear-stability analysis, we have performed simulations of perturbed evolution of the vortices in the framework of Eq. (2), using the split-step and D'yakonov methods [39,40]. We have chiefly employed a grid with 400×400 points and grid spacing $\Delta x=\Delta y=0.025$, the time step being $\Delta t=0.0005$. To ensure robustness of the algorithm, the results were compared to those produced with other values of $\Delta x=\Delta y$ and Δt. Initial random perturbations with a 5% relative amplitude were added in the simulations.

First, in Fig. 3, we display the perturbed evolution of a stable vortex with $S=3$ and $\mu=3.8$ ($N=69$), in terms of the density and phase. It exhibits rapid self-cleaning of the initially perturbed configuration, which is observed in the stability regions for all values of S.

In the instability region, the simulations display fragmentation of vortices, depending on the azimuthal index L of the respective leading perturbation eigenmode, as shown in Fig. 4. In this figure, the unstable vortex with $S=3$ breaks into four fragments, the corresponding dominant eigenmode indeed having $L=4$. Eventually, the fragments suffer intrinsic collapse.

Thus, the direct simulations corroborate the predictions of the linear-stability analysis for the trapped higher-order vorticities, including the fact that the splitting of unstable vortices is driven by dominant perturbation eigenmodes. It is noteworthy that, unlike the model with uniform attractive nonlinearity [21], here the stability and instability regions are not separated by a semistable one, where the vortex with $S=1$ periodically splits and recombines, keeping the vorticity intact.

Conclusions. We have introduced the model of a 2D trapped BEC with self-attractive nonlinearity acting in a spatially localized manner. Unlike the previously studied systems with uniform self-attraction, in which trapped vortices may be stable solely with topological charge $S=1$, the present setting gives rise to stability areas for multiple vortices, with $1\leqslant S\leqslant 6$.

TABLE II. Intervals ($N_{\min};N_{\max}$) in which the vortices are stable, at different values of S and b. For $S=1$, 3, and 4, only $N_{\max}$ is given, as $N_{\min}=0$ in these cases.

	$S=1$	$S=2$	$S=3$	$S=4$	$S=5$	$S=6$
$b=0.75$	11.0	(18 ; 20.3)	Unstable	Unstable	Unstable	Unstable
$b=1.0$	12.2	(0 ; 25)	Unstable	27.4	(42.5 ; 103)	(150 ; 172)
$b=1.5$	15.1	(0 ; 41)	101	237	(0 ; 550)	(0 ; 1274)
$b=2.0$	20.7	(0 ; 83)	299	1066	(0 ; 3829)	(2160 ; 13418)

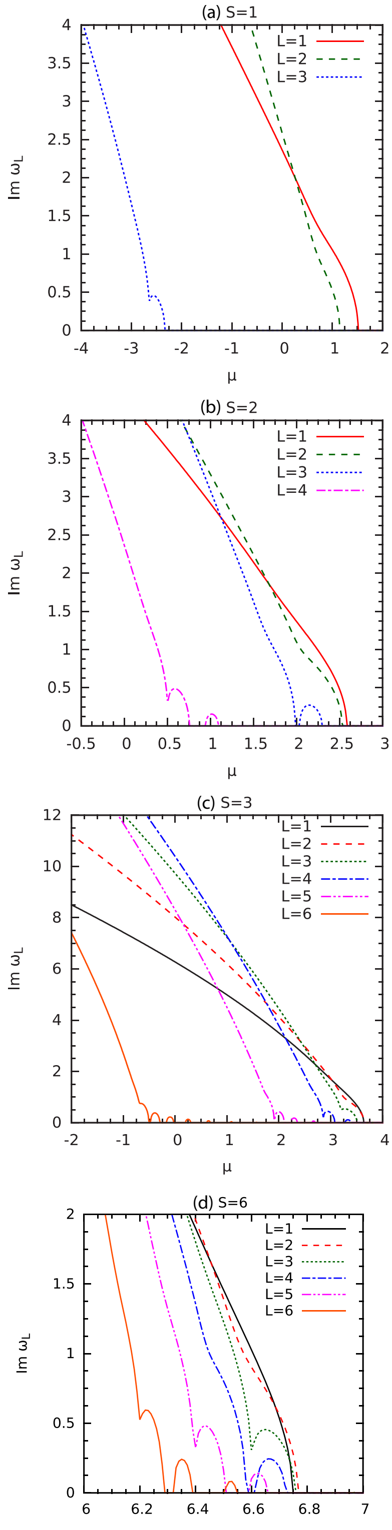

FIG. 2. (Color online) Imaginary part of the eigenfrequencies for different values of the azimuthal index L of perturbation modes for vortices with $S = 1$, 2, 3, and 6.

The largest radius of the nonlinearity localization which allows stabilization of higher-order vortices was estimated analytically and accurately found in numerical form, using linear-stability analysis and direct simulations. The number of atoms which can be held by a stable vortex sharply increases with S. Challenging problems for further analysis are the consideration of rotating multivortex complexes (cf. Ref. [31]) and generalization to the 3D setting.

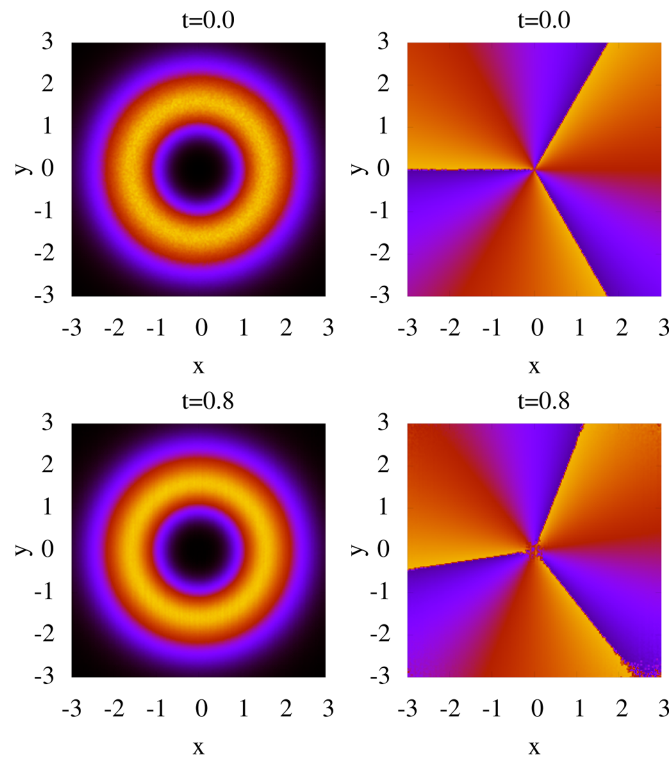

FIG. 3. (Color online) Evolution of the density and phase for a vortex with $S = 3$ and $\mu = 3.8$ ($N = 69$). Random perturbation, added at $t = 0$, is quickly eliminated by self-cleaning of the stable vortex, becoming invisible at $t = 0.8$, even though it is $\sim$0.1 of the characteristic diffraction time for the entire vortex. Here and in Fig. 4, in density profiles, brighter regions correspond to higher densities, and the phase varies from $-\pi$ to π for darker to brighter regions.

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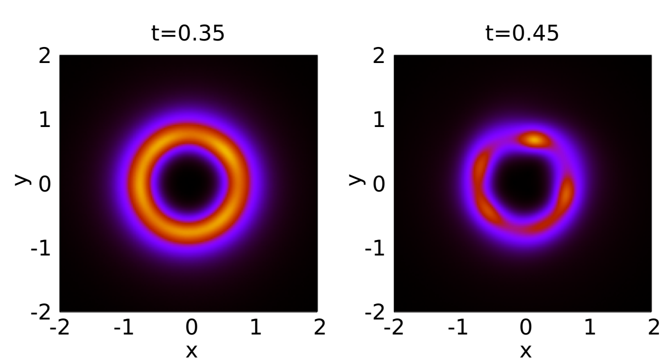

FIG. 4. (Color online) Perturbed evolution of the density field for an unstable vortex with $S = 3$ and $\mu = 0$ ($N = 90$).

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