Bose–Einstein condensation and non-extensive statistics for finite systems

E. Megías [a], V.S. Timóteo [b], A. Gammal [c], A. Deppman [c],*

[a] *Departamento de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Spain*
[b] *Grupo de Óptica e Modelagem Numérica, Faculdade de Tecnologia, GOMNI/FT, Universidade Estadual de Campinas, UNICAMP, Brazil*
[c] *Instituto de Física, Universidade de São Paulo, Brazil*

ABSTRACT

We study the Bose–Einstein condensation in non-extensive statistics for a free gas of bosons, and extend the results to the non-relativistic case as well. We present results for the dependence of the critical temperature and the condensate fraction on the entropic index, q, and show that the condensate can exist only for a limited range of q in both relativistic and non-relativistic systems. We provide numerical results for other thermodynamics quantities like the internal energy, specific heat and number fluctuations. We discuss the implications for high energy physics and hadron physics. The results for the non-relativistic case can be of interest in cold-atom systems.

© 2021 Elsevier B.V. All rights reserved.

1. Introduction

One of the most important results of the early Quantum Mechanics is the Bose–Einstein Condensate (BEC). It is a purely quantum effect, however its implications appear at scales similar to those of the classical systems. The description of this phenomenon results from the application of the standard Boltzmann–Gibbs (BG) Statistics to quantum systems [1,2].

Our knowledge on the statistical aspects of mechanical systems has evolved fast in the last decades. The introduction of non-additive entropy extended the reach of the statistical methods to new domains, where systems with complex structure, long-range interaction or following non-Markovian dynamics are found. Several new entropic formulas have been proposed [3], but the Tsallis's Entropy [4,5], S_q, is the most ubiquitous. The generalized entropy leads to a non-extensive thermodynamics that reduces to the standard Boltzmann–Gibbs theory when the entropic parameter, q, approaches the unit [6]. Although the reasons for the large number of systems following the Tsallis statistics are not yet clear, a recent result on the mechanism for the emergence of the non-additive entropy can give some clues on the subject [7,8].

It was shown that a fractal structure in the energy–momentum space, called thermofractal, can be present in any interaction system described in terms of the Yang–Mills fields [9]. Since this class of systems includes three of the four known interactions, it is clear that thermofractals can be found in many natural phenomena. Since the thermofractals exhibit the non-additive statistics properties, it may explain why systems following the S_q are so common.

While BEC has been exhaustively studied under the light of the BG Statistics [10–12], the same does not hold for the Tsallis Statistics. Apart from a few works on the S_q for quantum systems [13–19], and some applications for BEC, there is a

---

* Corresponding author.
  E-mail address: deppman@usp.br (A. Deppman).

lack of information on the behaviour of the condensates following the non-additive entropy. In the work by Chen et al. [20] the non-extensive BEC was considered in both relativistic and non-relativistic regimes. The approach developed in that work and in the present one are similar in many aspects, but we advance in the study considering the dependence on the number of particles, the behaviour of the functions with the parameter q, and the range in q one can obtain the condensate. One curious result obtained in that work is that the specific heat is not continuous at the critical temperature for some values of q. Here we show that for the number of particles below $N \sim 100$ the discontinuity is not observed, but we verified that it appears for large values of N (typically $N \gtrsim 100$). In this work we provide a detailed description of the non-extensive BEC, or qBEC. We investigate the constraints on the entropic parameter, q, for the formation of a condensate, and evaluate the behaviour of the system near the critical temperature.

Although S_q is associated with systems of interacting particles or strongly correlated systems, these aspects do not preclude the formation of BEC [10,21,22], even for systems with finite number of particles [11,12,23]. In this regard, it is interesting to observe that small systems will follow S_q, instead of BG statistics [24,25], nevertheless, most of the works are restricted to Boltzmann–Gibbs statistics, without considering the possibility of the existence of BEC in the non-extensive Tsallis Statistics applied to quantum systems.

Despite a relative lack of information on the non-extensive BEC, there are many applications of the concept in different areas, as in High Energy Physics and Hadron Physics, where the critical temperature of the phase-transition from the confined to the deconfined quark regimes are associated to the formation of a condensate [26,27], or in studies of the Neutron Star structure, where a phase-transition to the deconfined regime might appear [28]. A detailed study of the constrains for the formation of the condensate and its characteristics is needed. Since one of the motivations for the present work is the possibility of Bose–Einstein Condensate in the high energy collisions, we adopt the relativistic description, which can be straightforwardly restricted to the non-relativistic case, as commented in the present work.

In Section 1 we briefly review the basic concepts associated with BEC in the BG statistics, and in Section 2 we review the most important aspects of S_q for the present work. In Section 3 we describe the BEC in the non-extensive statistics, and in Section 4 we present our conclusions.In this work, we adopt the natural units, with $c = \hbar = k_B = 1$.

2. Bose–Einstein condensate in Boltzmann–Gibbs statistics

The thermodynamical potential for a relativistic gas of massless bosons in the Boltzmann–Gibbs statistics is given by [13]

$$\ln Z(T, V, \mu) = -\frac{V}{2\pi^2} \int_0^\infty d\varepsilon \, \varepsilon^2 \ln\left[1 - \exp[-\beta(\varepsilon - \mu)]\right] , \tag{1}$$

where $\beta = 1/T$ is the inverse of the temperature. Using the thermodynamical relation

$$\langle N \rangle = \beta^{-1} \frac{\partial}{\partial \mu} \ln Z \Big|_\beta , \tag{2}$$

we obtain the occupation number of particles,

$$N(T, V, \mu) = \frac{V}{2\pi^2} \int_0^\infty d\varepsilon \, \frac{\varepsilon^2}{\exp[\beta(\varepsilon - \mu)] - 1} . \tag{3}$$

The occupation number in the equation above diverges at the ground-state energy, $\varepsilon = 0$ in the limit $\mu \to 0$, therefore an infinitesimal vicinity around this value is excluded from the integration and added separately, and we get

$$N(T, V, \mu) \equiv N_0 + N_\varepsilon = \frac{1}{e^{-\beta\mu} - 1} + \frac{V}{2\pi^2} \mathcal{P} \int_0^\infty d\varepsilon \, \frac{\varepsilon^2}{\exp[\beta(\varepsilon - \mu)] - 1} , \tag{4}$$

with the first term being the number of particles in the ground-state, N_0, which is added separately. The symbol $\mathcal{P}$ in front of the integration sign is used to remember that the singular point is removed, and this integral corresponds to the number of particles in the excited states, N_ε. The integration can be computed numerically, and is associated to a family of integrals of the type

$$g_n(\mu) = \frac{1}{\Gamma(n)} \int_0^\infty d\varepsilon \, \frac{\varepsilon^{n-1}}{\exp[\beta(\varepsilon - \mu)] - 1} . \tag{5}$$

The analysis of the family of function described by the equation above shows that, for $n > 1$,

$$g_n(\mu) = \beta^{-n} \mathrm{Li}_n\left(e^{\beta\mu}\right) , \tag{6}$$

where Li is the polylogarithm function. In the case $\mu = 0$, one gets $g_n(0) = \beta^{-n}\zeta(n)$, where ζ is the zeta function. When the function is limited, the maximum of $g_n(\mu)$ happens for $\mu = 0$. Thus, N_ε remains finite and Eq. (4) gives the number of particles in the system. The ratio N_0/N is called condensate ratio, and is the most relevant quantity in the present work. When $\mu \to 0$, the maximum number of particles in the excited states is reached at a critical temperature, T_c, and is

$$N_\varepsilon(T) \leq N_{\varepsilon,\max}(T_c) \equiv \frac{VT_c^3}{\pi^2}\zeta(3) . \tag{7}$$

This is the maximum number of particles allowed in the excites states for the system at temperature T_c, and represents the total number of particles if $N_0 = 0$. The value for $\zeta(3)$ is finite, so the number of particles in the excited state, $N_\varepsilon(T)$ remains finite. Below the critical temperature, T_c, however, the number of particles allowed in the excited states, $N_\varepsilon(T)$, becomes smaller than the maximum number of particles at the critical temperature, so the excess of particles must be at the ground-state. If we consider that the total number of particles equals the maximum number of particles allowed at the critical temperature, from the equation above, we get

$$N = N_{\varepsilon,\max}(T_c) = \frac{VT_c^3}{\pi^2}\zeta(3), \tag{8}$$

therefore

$$T_c = \left(\frac{N}{V}\right)^{1/3}\frac{\pi}{[\pi\zeta(3)]^{1/3}}. \tag{9}$$

The ratio

$$\frac{N_0(T)}{N} = \frac{N - N_\varepsilon(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^3. \tag{10}$$

Note that we obtain cubic behaviour in the present case because we are considering a relativistic gas, what changes the phase-space topology. For a non-relativistic gas, we would have, instead of Eq. (10), a similar equation with power 3/2 in the temperature.

We will study also other quantities like the variance of the condensate population

$$\Delta N_0^2 = \beta^{-1}\frac{\partial}{\partial\mu}N_0 = \frac{e^{-\beta\mu}}{\left(e^{-\beta\mu} - 1\right)^2}, \tag{11}$$

the total energy of the system

$$U = -\frac{\partial}{\partial\beta}\ln Z\bigg|_\mu + \frac{\mu}{\beta}\frac{\partial}{\partial\mu}\ln Z\bigg|_\beta = \frac{V}{2\pi^2}\int_0^\infty d\varepsilon\,\frac{\varepsilon^3}{\exp[\beta(\varepsilon - \mu)] - 1} = \frac{3V}{\pi^2}g_4(\mu), \tag{12}$$

and the specific heat at constant volume

$$
\begin{aligned}
C_V = \frac{\partial U}{\partial T}\bigg|_{N,V} &= \frac{V}{2\pi^2}\beta^2\int_0^\infty d\varepsilon\,\varepsilon^3\left(\varepsilon - \mu + T\frac{\partial\mu}{\partial T}\right)\frac{\exp[\beta(\varepsilon - \mu)]}{(\exp[\beta(\varepsilon - \mu)] - 1)^2} \\
&= \frac{3V\beta}{\pi^2}\left[4g_4(\mu) - \left(\mu - T\frac{\partial\mu}{\partial T}\right)g_3(\mu)\right].
\end{aligned}
\tag{13}
$$

Observe that the condition that N is constant might lead to a temperature dependence of the chemical potential.

3. Bose–Einstein condensation in non-extensive statistics

Tsallis proposed a generalization of the Boltzmann–Gibbs statistics by introducing a new entropy formula, given by

$$S_q = k\frac{1 - \sum_i p_i^q}{q - 1} = -k\sum_i p_i^q \ln_q p_i, \tag{14}$$

where p_i is the probability, q is a parameter called entropic index, and

$$\ln_q z = \frac{z^{1-q} - 1}{1 - q} \tag{15}$$

is the q-logarithm function. The new entropic formula has two remarkable facts: it allows to obtain a complete thermodynamic description of the system through the Maximum-Entropy Principle; and it is non-additive.

The inverse of the q-logarithmic function is the q-exponential, that is common in the distributions of systems described by the non-extensive thermodynamics that is derived from the non-additive entropy.

3.1. Non-extensive thermodynamics

The non-extensive thermodynamics of quantum systems has been studied in [13,14]. It has been applied to investigate some systems, specially hadronic systems [29], with large implications in the study of, e.g., can be hadrons, Quark–Gluon plasma and neutron-stars. It is interesting to notice that the running-constant at the non-perturbative regime has been described by using the non-extensive statistics. The connections between the running-constant and the S_q has been demonstrated in [9]. Here, BEC in the non-extensive statistics, using as starting point is the non-extensive quantum ideal gas, which was described in Ref. [13].

The non-extensive entropy is given by the standard thermodynamics relation

$$S = -\beta^2 \frac{\partial}{\partial \beta} \left(\frac{\ln Z_q}{\beta} \right) \Bigg|_\mu , \tag{16}$$

where $\beta = 1/(kT)$, with k being the Boltzmann constant. For the non-extensive ideal gas of bosons or fermions, the thermodynamical potential is given by

$$\ln Z_q(V, T, \mu) = -\xi V \int \frac{d^3 p}{(2\pi)^3} \sum_{r=\pm} \Theta(rx) \ln_q^{(-r)} \left(\frac{e_q^{(r)}(x) - \xi}{e_q^{(r)}(x)} \right) . \tag{17}$$

In the equations above, we have used the q-exponential functions

$$\begin{cases} e_q^{(+)}(x) = [1 + (q-1)x]^{\frac{1}{q-1}} , & x \geq 0 \\ e_q^{(-)}(x) = [1 - (q-1)x]^{-\frac{1}{q-1}} , & x < 0 \end{cases} , \tag{18}$$

and the q-logarithm functions, $\ln_q^{(+)}(x)$ and $\ln_q^{(-)}(x)$, defined as the inverse function of, respectively, $e_q^{(+)}(x)$ and $e_q^{(-)}(x)$. The parameter ξ distinguishes the cases of bosons, with $\xi = 1$, and fermions, with $\xi = -1$. Here we are interested in the first case, so we fix $\xi = 1$. Also, we will deal with positive energies, $\varepsilon \geq 0$, and negative chemical potentials, $\mu \leq 0$, so $x := \beta(\varepsilon - \mu) \geq 0$, where ε is the single-particle state energy. Then, the relevant q-functions will be $\ln_q^{(-)}(x)$ and $e_q^{(+)}(x)$.

The properties of the q-exponential function have been investigated in many works, and many of those properties are embedded in the so-called q-calculus [30]. An important property is the duality $q \leftrightarrow q' = 2 - q$, which is very common to be observed in non-extensive systems. This duality was investigated in Ref. [31] in the context of the q-reflection transformation that changes $q = 1 + \delta$ to $q' = 1 - \delta$, and it was shown that this transformation is isomorphic to a reflection of the function domain, $x \to -x$. In these regards, it is interesting to mention that for quantum relativistic systems in the non-extensive statistics, where particles pair creation is allowed, a complete description of the system in thermodynamical terms is possible only with the inclusion of distributions with both q and $q' = 2 - q$ [32].

3.2. Bose–Einstein condensation and non-extensive effects

Using the thermodynamical relation

$$\langle N \rangle = \beta^{-1} \frac{\partial}{\partial \mu} \ln Z_q \Big|_\mu , \tag{19}$$

we obtain the single-particle state occupation number for a bosonic gas,

$$n_q^{(+)}(\varepsilon, \beta, \mu) = \left[e_q^{(+)}[\beta(\varepsilon - \mu)] - 1 \right]^{-q} . \tag{20}$$

We will investigate in what conditions the singularity corresponding to the Bose–Einstein condensation is present in the formula above. Let ε_c be the single-particle energy that is the closest to the chemical potential value, μ. Following the standard approach, we consider this case as separate from the other states, and its occupation number will be

$$n_q^0(\varepsilon_c, \beta, \mu) = \left[e_q^{(+)}[\beta(\varepsilon_c - \mu)] - 1 \right]^{-q} , \tag{21}$$

while the number of particles in the other states are

$$n_q^\varepsilon(\varepsilon, \beta, \mu) = \left[e_q^{(+)}[\beta(\varepsilon - \mu)] - 1 \right]^{-q} . \tag{22}$$

Hereby, the total number of particles is

$$N_q \equiv N_q^0 + N_q^\varepsilon = \left[e_q^{(+)}[\beta(\varepsilon_c - \mu)] - 1 \right]^{-q} + \frac{V}{2\pi^2} \int_0^\infty d\varepsilon \, \varepsilon^2 \left[e_q^{(+)}[\beta(\varepsilon - \mu)] - 1 \right]^{-q} . \tag{23}$$

If we consider $\mu \to 0$, the singularity in the occupation number corresponds to the ground-state, $\varepsilon_c = 0$.

A careful functional analysis must be carried out in order to verify if the condensate can be formed in the non-extensive statistics, and what are the conditions and the qBEC behaviour in the positive case. For simplicity we introduce $x = \beta(\varepsilon - \mu)$ in the following analysis. The expression for the number of particles in the excited states becomes

$$N_q^\varepsilon = \frac{V}{\pi^2} \beta^{-3} \zeta_q(\mu\beta) , \tag{24}$$

where[1]

$$\zeta_q(\mu\beta) = \frac{1}{2} \int_{-\mu\beta}^\infty dx \, (x + \mu\beta)^2 \left[e_q^{(+)}(x) - 1 \right]^{-q} . \tag{25}$$

---

[1] The definition of Eq. (25) corresponds to values $q \geq 1$. The definition of $\zeta_q(\mu\beta)$ for $q < 1$ includes an upper limit in the integral: $\int_{-\mu\beta}^\infty dx \to \int_{-\mu\beta}^{1/(1-q)} dx$.
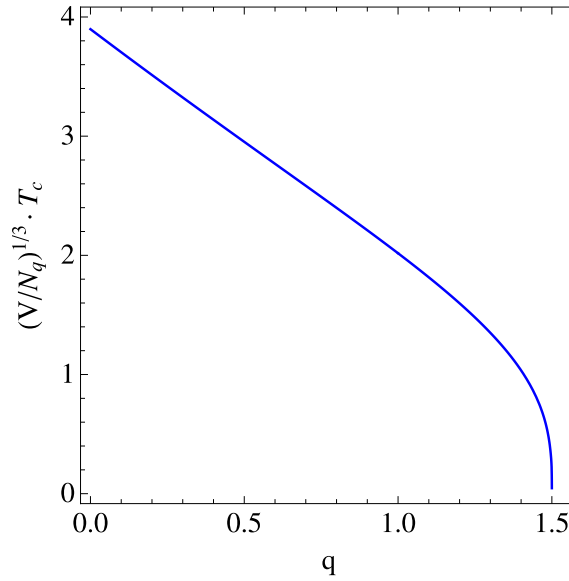
Fig. 1. Critical temperature ($\times (V/N_q)^{1/3}$) as a function of the entropic index q. Notice that the curve is independent of the values of V and N_q.

Notice that the maximum number of particles allowed at the critical temperature turns out to be

$$N_{q,\text{max}}^{\varepsilon}(T_c) = \frac{VT_c^3}{\pi^2} \zeta_q(0) \,, \tag{26}$$

and therefore the critical temperature is

$$T_c = \left(\frac{N_q}{V}\right)^{1/3} \frac{\pi}{\left[\pi \zeta_q(0)\right]^{1/3}} \,. \tag{27}$$

This expression is the generalization of Eq. (9) according the non-extensive statistics. Observe that $\zeta_{q=1}(0) = \zeta(3)$. In Fig. 1 we show the behaviour of the product $(V/N_q)^{1/3}T_c$, which depends only on the entropic parameter q. The product decreases almost linearly up to the vicinity of the critical value $q_c = 3/2$. We will show below that this represents the maximum value for the formation of the BEC in non-extensive systems, and that at this limit, the condensate is formed only at temperatures near $T = 0$, and the low temperature is a manifestation of a phenomenon that will be investigated in details here, the fact that systems with higher entropic parameters resist to the formation of the condensate, while systems with low values of q are more favourable to the formation of the condensate.

Before studying the physical properties of the qBEC, we will study what are the conditions for it to be formed. The result obtained for the critical temperature already suggests some critical value, and we will show here how it arises in the non-extensive statistics. Considering the q-exponential function in its first order in the Taylor's expansion around the value $x = 0$, the integral above for $\mu\beta = 0$ becomes ($\zeta_q \equiv \zeta_q(0)$)

$$\zeta_q = \frac{1}{2} \int_0^{\infty} dx\, x^{2-q} dx \,, \tag{28}$$

which remains finite in the limit $x \to 0$ if $q < 3$. But we have to investigate the limit $x \to \infty$. In this case we have

$$\int^{x_+} dx\, x^2 \left[e_q^{(+)}(x) - 1\right]^{-q} \to \frac{(q-1)^{-\frac{1}{q-1}}}{3-2q} x_+^{3-\frac{q}{q-1}} \qquad (q > 1) \,, \tag{29}$$

which remains finite in the limit $x_+ \to \infty$ only if $q < 3/2$. Therefore, the number of particles in the excited states will remain finite if $1 < q < 3/2$. In fact, the case $q = 1$ leads also to finite values, as this corresponds to the Boltzmann–Gibbs condensate which was addressed in Section 2. Let us mention that in the case $q < 1$, the integral in the lhs of Eq. (29) has the upper limit $x_+ = 1/(1-q)$, and it turns out to be finite for any value of $q < 1$. Then, the range of values of q in which N_q^{ε} is finite can in fact be extended to any value $q < 3/2$. These results explain the limiting value for the parameter q observed in Fig. 1. Therefore, the qBEC can be obtained only if $q < 3/2$.

Observe that ζ_q is independent of the temperature, so we can calculate the condensate ratio in a way similar to the one used in the BG condensate. We observe that there is a critical temperature for which the number of particles in
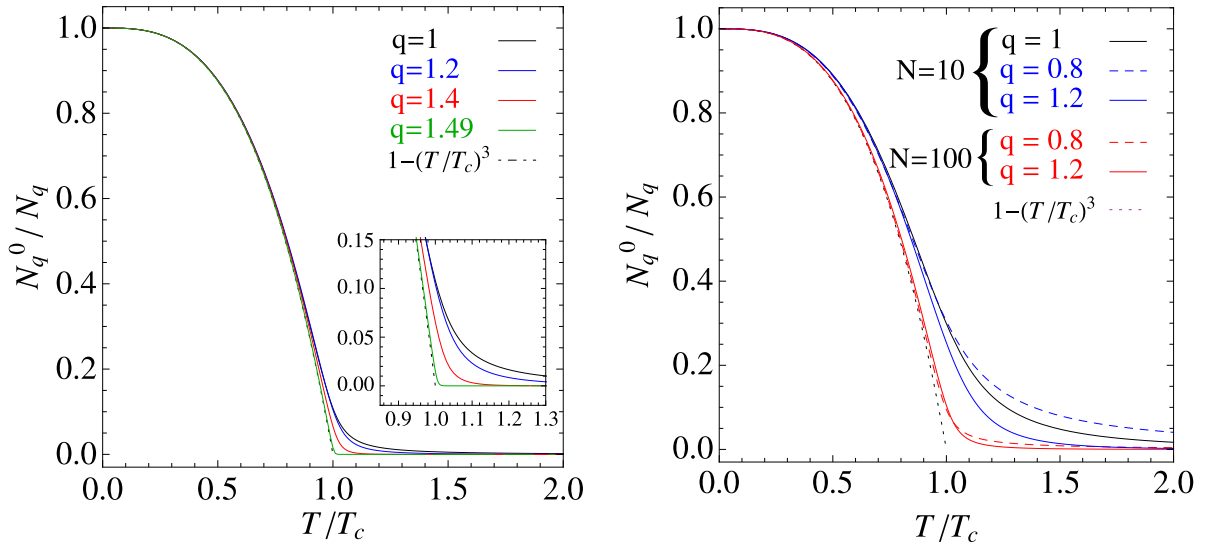
Fig. 2. Bose–Einstein condensation in the non-extensive statistics. Left panel: plot for fixed value of the number of particles, $N = 100$ and different values of the entropic index, q. Right panel: plot for fixed values of the entropic index, $q = 0.8$ and 1.2, and different number of particles.

the condensate, N_q^0, is still much smaller than the total number N_q, and the number of particles in the excited states is maximum. Below this temperature, any new particle will populate the condensate. In this case,

$$\frac{N_q^0}{N_q} = 1 - \frac{N_q^\varepsilon}{N_q} = 1 - \left(\frac{T}{T_c}\right)^3 . \tag{30}$$

The expression for the number of particles in the excited state is, now, slightly different that in Eq. (3), since it gives, in the non-extensive case,

$$N_q^\varepsilon = \frac{V}{2\pi^2} \int_0^\infty d\varepsilon \, \varepsilon^2 \left[e_q^{(+)}[\beta(\varepsilon - \mu)] - 1 \right]^{-q} . \tag{31}$$

Thus, the non-extensive Bose–Einstein condensate presents the same behaviour with the temperature as the Boltzmann–Gibbs condensate. This result was verified by numerical calculations (see Section 3.4), as shown in Fig. 2, where we see that the condensate fraction approaches the expected behaviour as the number of particles increases, for different values of q in the allowed range.

In the left and right panels of Fig. 2 we plot the result of N_q^0/N_q as a function of T/T_c performed by using this numerical procedure. We observe in the left panel that, as $q \rightarrow 3/2$, the ratio N_q^0/N_q approaches the critical line expected in the Boltzmann–Gibbs Statistics, independently of the number of particles or temperature. In the right panel of this figure it is observed that the same effect is obtained when increasing the number of particles independently of the value of q.

The results shown in Figs. 1–2 evidence an interesting behaviour of the qBEC that cannot be observed in the Boltzmann–Gibbs BEC. While in Fig. 1 we observed the resistance of the system to form the condensate as q increases, which is manifested in the lower critical temperatures in those cases. However, in Fig. 2 we notice a behaviour that, at first sight, seems to contradict the previous conclusion, since they show that the limiting behaviour of the qBEC formation is attained by the systems with higher values of q before those with lower values. This contradiction is not real, though, since in the latter figures the condensate fraction is plotted as a function of the ratio T/T_c.

The conclusions we can draw from these initial results are that, while the qBEC resistance for the condensate formation increases with q, the phase-transition to the condensate is sharper for systems with large values of q than for those system with lower values of q. To investigate this behaviour in more details, we will study the dependence of the number of particles in the first excited state on the temperature.

We will show that the analysis of the fraction of particles in the first excited states confirms the conclusions obtained above. A few modifications in our method are necessary, so let us describe them before turning our attention to the physical aspects of the problem. While in Eq. (23) the system is supposed to have a continuum of states, one can obtain a discretization of the energy levels when considering it inside a large cubical box of length L. Then, the energy levels of relativistic massless particles is

$$E_{n_x,n_y,n_z} = \frac{\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}, \qquad n_x, n_y, n_z \geq 1 . \tag{32}$$

The results obtained with the method described above are plotted in Fig. 3. We see that there is a peak in the ratio N_q^1/N_q as a function of the relative temperature, T/T_c. The presence of the peak is observed for all values of q below q_c,
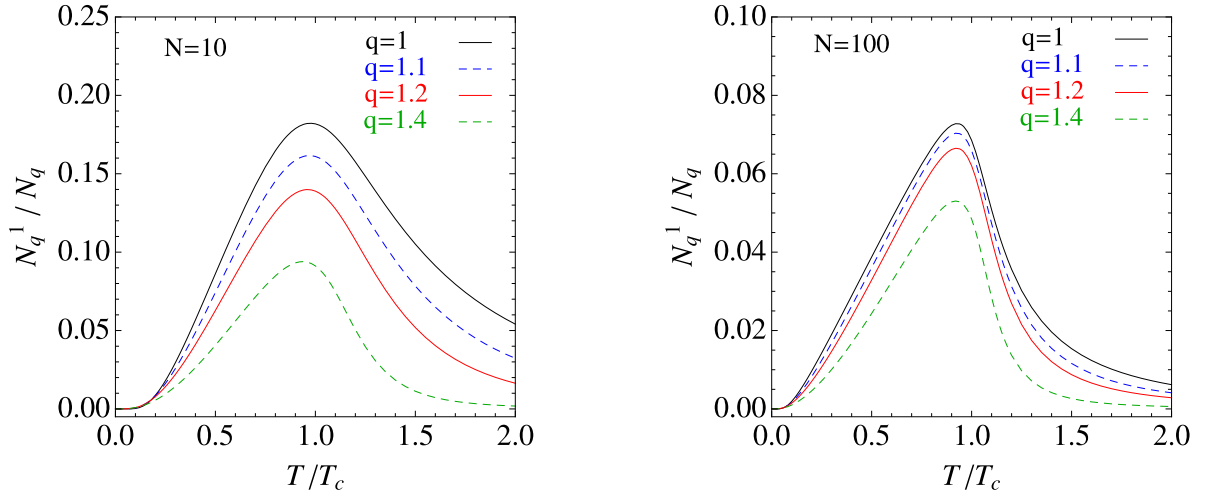
Fig. 3. Fraction of particles in the first excited state within the non-extensive statistics for fixed values of the number of particles, $N = 10$ (left panel) and $N = 100$ (right panel), and different values of the entropic index, q.
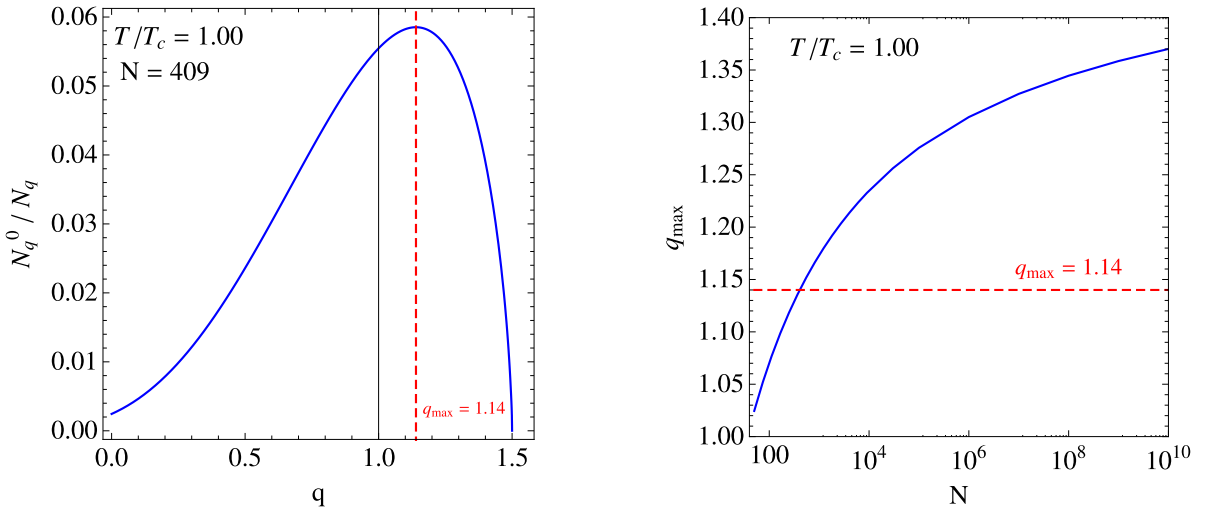
Fig. 4. Left panel: fraction of particles in the condensate within the non-extensive statistics as a function of q. The maximum value of N_q^0/N_q is obtained at q_{max}. Right panel: behaviour of q_{max} as a function of N_q. The value $q_{max} = 1.14$ is obtained for $N = 409$. It is shown as dashed (red) lines in both panels the curve $q_{max} = 1.14$. We have considered $T/T_c = 1$ in both panels.

and at a position near the critical temperature. As the number of particles increases, the peak becomes narrower. A clear dependence on the value of q is observed, and as q increases the relative number of particles in the first excited state tends to decrease. The shape of the curves are similar in all cases, including for $q = 1$ that corresponds to the BG condensate.

The reduction of the number of particles for higher values of q is associated with the sharper phase-transition discussed above, and results from the fact that a larger fraction of the particles are in the ground-state. This aspect is already present in Fig. 2, where the smaller tails at the right side of the critical temperature indicates the predominance of the condensate for those systems with large q. Therefore, the results of the analysis of the first excited state confirms the conclusions obtained above.

Another interesting aspect of the system that can be observed only in the non-extensive condensate can be observed in Fig. 3. Comparing the left and right panels of the figure, we note that the distributions with $q = 1.0$, $q = 1.1$ and $q = 1.2$, that are clearly separated in the case of $N = 10$, becomes very similar for $N = 100$. This result indicates that the relative contributions of the particles in the excited states changes with the number of particles. We will investigate further this aspect of the qBEC.

The behaviour of the part of the system at the excited states near the critical temperature is complex, therefore we will use some approximations to evaluate how the number of particles in the excited states vary with $0 < q < 3/2$. As

discussed above, we can divide the integration in two regions, small x and large x, and the approximations

$$\zeta_q \simeq \zeta_{q\,1}(x_1) + \zeta_{q\,2}(x_2) \tag{33}$$

with

$$\begin{cases} \zeta_{q\,1}(x_1) = \frac{1}{2} \int_0^{x_1} x^{2-q} dx \\ \zeta_{q\,2}(x_2) = \frac{1}{2}(q-1)^{-\frac{q}{q-1}} \int_{x_2}^{\infty} x^{2-\frac{q}{q-1}} dx \end{cases} \tag{34}$$

are valid for appropriately chosen x_1 and x_2. We will assume that there is a single value $x_1 = x_2 = X$ for which the approximations above can be considered valid, where X is to be determined. In the range of values for q mentioned above the integrations can be easily performed, and we get

$$\begin{cases} \zeta_{q\,1}(X) = \frac{1}{2} \frac{1}{3-q} X^{3-q} \\ \zeta_{q\,2}(X) = \frac{1}{2} \frac{(q-1)^{-\frac{1}{q-1}}}{3-2q} X^{2-\frac{1}{q-1}} \end{cases} . \tag{35}$$

We choose X such that the integrands of the functions defined in Eq. (34) are continuous. This condition is satisfied only if

$$X = (q-1)^{-\frac{1}{2-q}} . \tag{36}$$

With the approximations made above, we obtain the ζ_q function

$$\zeta_q = \frac{q(2-q)}{2(3-q)(3-2q)} (q-1)^{-1-\frac{1}{2-q}} . \tag{37}$$

The number of particles in the excited states is, therefore, given by

$$N_q^\varepsilon = \frac{V}{\pi^2} \beta^{-3} \frac{q(2-q)}{2(3-q)(3-2q)} (q-1)^{-1-\frac{1}{2-q}} . \tag{38}$$

The approximation used to obtain this analytical result works better for values of q close to $3/2$, while for $q \lesssim 1.3$ there are some deviations. The equation above indicates that the number of particles would diverge when q tends to $q = 3/2$.

The critical value, $q_c = 3/2$, appear also in the analysis of the condensate fraction as a function of q. In Fig. 4 (left) we observe a peak in the curve at a position q_{max}. which depends on the number of particles in the system. The behaviour of q_{max} with N is shown in Fig. 4 (right), and we note its continuous increase with the number of particles, asymptotically approaching the value $q_c = 3/2$. This plot, indeed, is related to that in Fig. 1.

The critical value, q_c, is associated with the system for which the condensate can be formed just at $T = 0$. On the other hand, the N particle system with $q = q_{max}(N)$ is the one with the sharpest transition to the condensate regime, as evidenced by the maximum ratio N_q^0/N_q.

3.3. Total energy, specific heat and number fluctuations

The aspects of the qBEC analysed so far do not consider the energy of the system. It turns out that the energy imposes some additional constraints on the non-extensive system. The total energy of the system is obtained by the standard relation

$$U_q = -\frac{\partial}{\partial \beta} \ln Z_q \Big|_\mu + \frac{\mu}{\beta} \frac{\partial}{\partial \mu} \ln Z_q \Big|_\beta \tag{39}$$

resulting in

$$U_q = \frac{V}{2\pi^2} \int_0^\infty d\varepsilon \, \varepsilon^3 \left[e_q^{(+)}[\beta(\varepsilon - \mu)] - 1 \right]^{-q} . \tag{40}$$

Notice that the integrand now has a term p^3, even though we are in the relativistic regime. This is because we are now calculating the total energy, while in the previous sections we were dealing with the total number of particles. By using a matching procedure similar to the one presented for N_q^ε, the total energy can be approximated by

$$\begin{aligned} U_q &= \frac{V\beta^{-4}}{2\pi^2} \int_0^\infty dx \, x^3 \left[e_q^{(+)}(x) - 1 \right]^{-q} \\ &= \frac{V\beta^{-4}}{2\pi^2} \left[\int_0^X dx \, x^{3-q} + (q-1)^{-\frac{q}{q-1}} \int_X^\infty dx \, x^{3-\frac{q}{q-1}} \right] , \end{aligned} \tag{41}$$

for X given by Eq. (36). The first integral in the last line of Eq. (41) converges to a finite value for $q < 4$, but the second integral converges only if $q < 4/3$. The result for U_q is

$$U_q = \frac{V\beta^{-4}}{2\pi^2} \frac{q(2-q)}{(4-q)(4-3q)} (q-1)^{-1-\frac{2}{2-q}} , \tag{42}$$
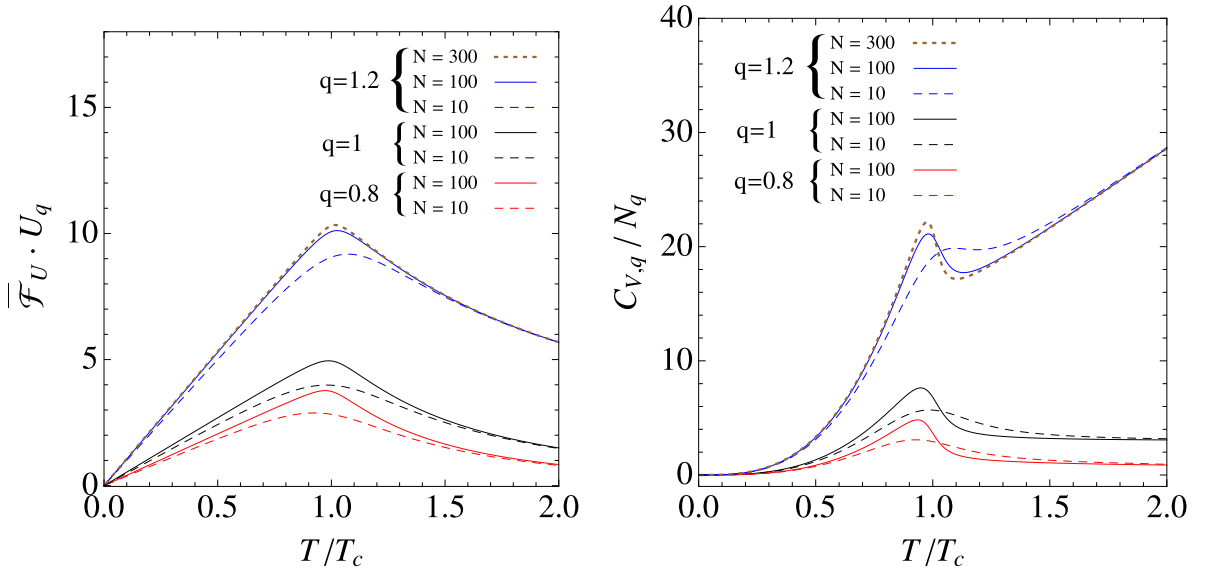
Fig. 5. Total energy, normalized by the factor $\bar{\mathcal{F}}_U := N_q^{-1}(V/N_q)^{1/3}(T_c/T)^3$ (left panel), and specific heat at constant volume ($\times N_q^{-1}$) (right panel), in the non-extensive statistics. We display the results for fixed values of the entropic index, $q = 0.8$ and 1.2, and different number of particles, $N = 10$, 100 and 300. It is displayed also the results in BG statistics in both panels.

which turns out to be a good approximation for $1.3 \lesssim q < 4/3$. Therefore, in the limit for q found in the analysis of the condensate fraction, those systems with $q > 4/3$ present an infinite energy, despite the finite number of particles. The conclusion is that they do not represent a physical system.

The energy of the system for $q = 0.8$ and $q = 1.2$, as a function of T/T_c, is shown in the left panel of Fig. 5. We observe a linear behaviour for temperatures below T_c, which is a consequence of the formation of the condensate since the number of particles in the ground-state is proportional to T. The energy of the system with $q = 0.8$ remains below that for $q = 1$, while for $q = 1.2$ the energy remains systematically above the Boltzmann–Gibbs cases.

The system with large q presents a more pronounced change at the critical temperature than those with small q. This aspect can be appreciated also by observing the behaviour of the specific heat at constant volume, $C_{V,q}$, that is calculated by

$$C_{V,q} = \frac{\partial U_q}{\partial T}\bigg|_{N_q,V} = \frac{V}{2\pi^2}\beta^2 q \int_0^\infty d\varepsilon\, \varepsilon^3 \left(\varepsilon - \mu + T\frac{\partial\mu}{\partial T}\right) \frac{e_q^{(+)}[\beta(\varepsilon-\mu)]^{2-q}}{\left(e_q^{(+)}[\beta(\varepsilon-\mu)]-1\right)^{q+1}}. \tag{43}$$

In the right panel of Fig. 5 we see that $C_{V,q} \propto T^3$ up to the critical temperature, and there is a change of regime for $T \gtrsim T_c$ reflecting the modification in the system structure. There is a clear dependence on q in both U_q and $C_{V,q}$. We do not observe any discontinuity in the specific heat at the critical point for finite values of N, contrary to the results obtained in [20] for the relativistic case. However, we have checked that a gap is formed for number of particles larger than 100 with $\Delta C_{V,q} = C_{V,q}(T \gtrsim T_c) - C_{V,q}(T \lesssim T_c) < 0$, and eventually this becomes a discontinuity in the limit $N \to \infty$. The main difference between our approach and that followed by Chen et al. apart from calculation procedures slightly different, relies in the occupation number, with ours corresponding to a power q of the occupation number used in that work. In this regard, we recall that powers in probability densities are a consequence of the Tsallis entropy, and in our case, it was obtained from a complete thermodynamical description of a relativistic gas in the Tsallis statistics [13].

Finally, we calculate the variance of the condensate population in the non-extensive statistics for fixed total number of particles, given by

$$\Delta N_q^{0\,2} = \beta^{-1}\frac{\partial}{\partial\mu}N_q^0 = q\frac{e_q^{(+)}[-\beta\mu]^{2-q}}{\left(e_q^{(+)}[-\beta\mu]-1\right)^{q+1}}. \tag{44}$$

This is plotted in Fig. 6 for fixed values of the entropic index. Notice that the variance tends to decrease with the value of q. This is an interesting quantity, since it can be measured experimentally [33].

We note that the non-extensive non-relativistic case can be treated in a similar way, but a few results will be different. The behaviour of the condensate fraction, as given by Eq. (10) will depend on $(T/T_c)^{3/2}$. The range of value for the entropic parameter, q, where the BEC can be obtained will be $0 < q < 3$, while the energy of the system will be finite if $0 < q < 5/3$. These differences between the relativistic and non-relativistic are due to the topology of the phase-space.
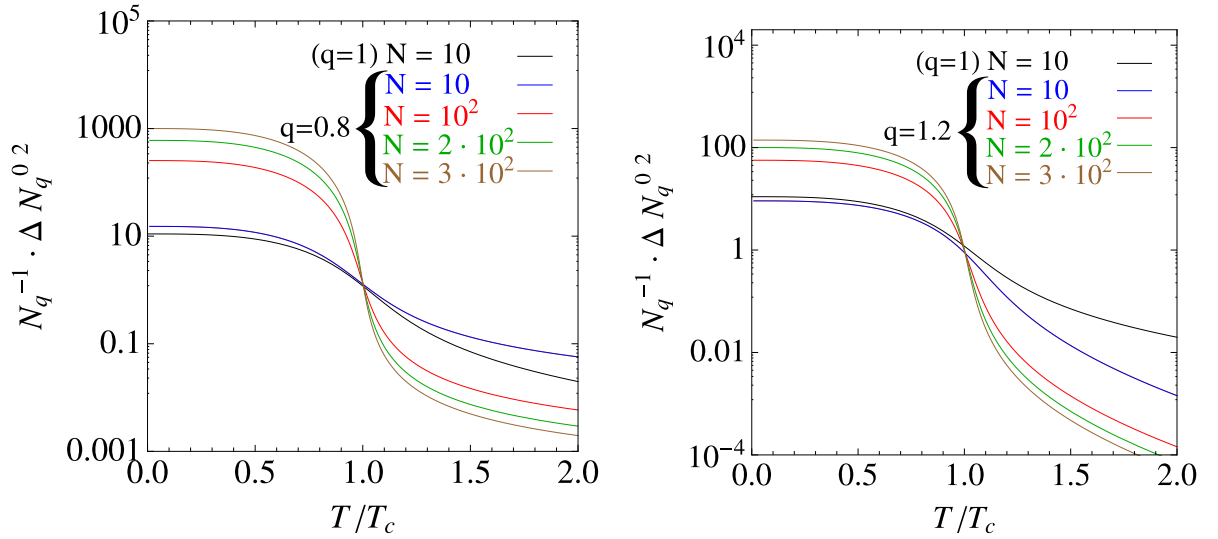
Fig. 6. Variance of the condensate population in the non-extensive statistics for fixed values of the entropic index, $q = 0.8$ (left panel) and $q = 1.2$ (right panel), and different number of particles.

3.4. Numerical strategy

Apart from these analytical considerations, we have performed a numerical study of Bose–Einstein condensation. The numerical calculations are performed with the following steps:

1. Given a value $\bar{N}_q$ for the total number of particles, the critical temperature T_c is then computed as the value of temperature such that $\bar{N}_q = N_{q,\max}^{\varepsilon}(T_c)$. This leads to a dependence $T_c \propto (\bar{N}_q/V)^{1/3}$, a property that was already obtained both in BG and in Tsallis statistics, cf. Eqs. (9) and (27).
2. For the same value of $\bar{N}_q$ and for a given temperature T, usually different from T_c, it is computed the value of the chemical potential μ such that $\bar{N}_q = N_q(T, \mu) = N_q^0(T, \mu) + N_q^{\varepsilon}(T, \mu)$.
3. We evaluate the ratios $N_q^0(T, \mu)/\bar{N}_q$ and T/T_c at temperature T.

Let us mention that the curves of $N_q^0/\bar{N}_q$ and $N_q^1/\bar{N}_q$ as a function of T/T_c are independent on the value of V, and the same can be said for the variance $\Delta N_q^{0\,2}$, the specific heat $C_{V,q}$ and the product $V^{1/3}U_q$.

4. Conclusions and outlook

We studied the Bose–Einstein Condensate in the non-extensive statistics (qBEC), contributing to complement the lack of information on those systems. We calculated the critical temperature and the condensate fraction. These quantities present a dependence on the entropic parameter, q. We show that the qBEC can exist in non-extensive system only for $q < q_c$ with $q_c = 3/2$.

At the critical value, q_c, the critical temperature is null. As q decreases from its maximum value, the critical temperature shows, initially, a fast increase, and then, around $q = 4/3$, it increases linearly as q decreases. This behaviour was interpreted as a resistance of the system to the formation of the condensate. The large is the entropic parameter, the stronger is the resistance.

On the other hand, those systems with larger values for q present a sharper phase-transition. This was evidenced by the behaviour of the condensate fraction at the critical temperature. The dependence of the condensate fraction with the entropic parameter was investigated in details, and we study it for all values of $0 < q < 3/2$.

We studied the energy and the specific heat of the qBEC, and showed that the energy presents a divergence for $q > 4/3$. Therefore, physical qBEC can exist only in the range $0 < q < 4/3$. In the condensate regime, the specific heat is constant for all values of q, but its value is q-dependent. This dependence arises from the fact that the condensate fraction depends on the value q.

We analysed the variance of the number of particles.

The possibility of formation of a condensate phase in the hadronic systems have been postulated in several works [27, 34–36]. Investigations on the High Energy Physics distributions show that, for hadronic systems, $q \sim 1.14$, in good agreement with the theoretical prediction [8,9]. The results obtained here show that the QCD allows for the formation of a condensate regime of the gluonic field. The implications of this results go beyond the high energy collisions, reaching

the hadron [26,37,38] and neutron star structures [28,39–42]. It is remarkable that the results on the fractal structure of Yang–Mills fields include systems governed by electrodynamic interaction, for which one expects $q < 1$, so effects of the qBEC can be investigated in cold atom condensate as well. The observation of power-law distributions and chaotic behaviour in Bose–Einstein condensate with vertex [43–45] can be indicative of non-extensive behaviour in those systems. In these regards, we commented about the extension of the results obtained here to the non-relativistic case.

The transition behaviour observed in the Bose–Einstein condensates when the temperature changes is of the same nature of the behaviour of the quark condensates when the coupling changes (in a model with only contact interactions). But there is a difference though: quark models are handled with field theory techniques. We plan to perform a similar study for hadronic condensates trying to connect the deviations from the extensive statistics, measured by q, to some sort of effective interaction added to the system.

Finally, let us point out that while we have been mostly interested in possible applications of BEC to high energy collisions, hadron physics and the Quark–Gluon plasma, and subsequently we have mostly studied the case of relativistic particles with negligible mass, there are other physical systems affected by BEC in which relativistic effects are important. This is the case, for instance, of the BEC of spin triplet excitations (magnons) in $TlCuCl_3$, first discovered in [46]. It was stressed in later publications [47,48] the need of relativistic dispersion relations to correctly describe the experimental data also for not extremely low temperatures. In principle, the present non-extensive statistics study could be extended to investigate the magnons BEC. Notice, however, that such quantum spin systems are governed by the spin–spin interaction, instead of the strong interaction, which might affect at least the range of values for q that turns out to be relevant in this case. This study might be relevant also for the understanding of the underlying dynamics of non-extensive statistics in atomic systems.

CRediT authorship contribution statement

E. Megías: Idealized, Developed the formalism, Discussion and analysis. **V.S. Timóteo:** Idealized, Discussion and analysis. **A. Gammal:** Discussion and analysis. **A. Deppman:** Idealized, Developed the formalism, Discussion and analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.physa.2021.126440. Supplementary material is available online, showing additional plots of quantities calculated in the present work.

References

[1] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, S. Stringari, Theory of Bose-Einstein condensation in trapped gases, Rev. Modern Phys. 71 (1999) 463–512, http://dx.doi.org/10.1103/RevModPhys.71.463.
[2] K. Huang, Introduction To Statistical Physics, CRC Press, Boca Raton, 2010.
[3] C. Beck, E. Cohen, Superstatistics, Physics A 322 (1–4) (2003) 267–275, http://dx.doi.org/10.1016/S0378-4371(03)00019-0.
[4] C. Tsallis, Possible generalization of the Boltzmann–Gibbs statistics, J. Stat. Phys. 52 (1–2) (1988) 479–487, http://dx.doi.org/10.1007/BF01016429.
[5] C. Tsallis, Introduction To the Non-EXtenssive Statistical Mechanics, Springer, New York, 2009.
[6] E. Curado, C. Tsallis, Generalized statistical-mechanics - connections with thermodynamics, J. Phys. A: Math. Gen. 24 (2) (1991) L69–L72, http://dx.doi.org/10.1088/0305-4470/24/2/004.
[7] A. Deppman, Thermodynamics with fractal structure, Tsallis statistics, and hadrons, Phys. Rev. D 93 (5) (2016) http://dx.doi.org/10.1103/PhysRevD.93.054001.
[8] A. Deppman, E. Megías, D.P. Menezes, Fractal structures of yang–mills fields and non-extensive statistics: Applications to high energy physics, Physics 2 (3) (2020) 455–480, http://dx.doi.org/10.3390/physics2030026.
[9] A. Deppman, E. Megias, D.P. Menezes, Fractals, nonextensive statistics, and QCD, Phys. Rev. D 101 (3) (2020) http://dx.doi.org/10.1103/PhysRevD.101.034019.

[10] V. Bagnato, D. Pritchard, D. Kleppner, Bose–Einstein condensation in an external potential, Phys. Rev. A 35 (10) (1987) 4354–4358, http://dx.doi.org/10.1103/PhysRevA.35.4354.
[11] M. Mewes, M. Andrews, N. vanDruten, D. Kurn, D. Durfee, C. Townsend, W. Ketterle, Collective excitations of a Bose–Einstein condensate in a magnetic trap, Phys. Rev. Lett. 77 (6) (1996) 988–991, http://dx.doi.org/10.1103/PhysRevLett.77.988.
[12] V. Bagnato, D. Kleppner, Bose–Einstein condensation in low-dimensional traps, Phys. Rev. A 44 (11) (1991) 7439–7441, http://dx.doi.org/10.1103/PhysRevA.44.7439.
[13] E. Megias, D.P. Menezes, A. Deppman, Non extensive thermodynamics for hadronic matter with finite chemical potentials, Physica A 421 (2015) 15–24, http://dx.doi.org/10.1016/j.physa.2014.11.005.
[14] S. Mitra, Thermodynamics and relativistic kinetic theory for q-generalized Bose–Einstein and Fermi-Dirac systems, Eur. Phys. J. C 78 (1) (2018) http://dx.doi.org/10.1140/epjc/s10052-018-5536-3.
[15] A. Rajagopal, R. Mendes, E. Lenzi, Quantum statistical mechanics for nonextensive systems: Prediction for possible experimental tests, Phys. Rev. Lett. 80 (18) (1998) 3907–3910, http://dx.doi.org/10.1103/PhysRevLett.80.3907.
[16] E. Lenzi, R. Mendes, A. Rajagopal, Quantum statistical mechanics for nonextensive systems, Phys. Rev. E 59 (2, A) (1999) 1398–1407, http://dx.doi.org/10.1103/PhysRevE.59.1398.
[17] O. Savchuk, Y. Bondar, O. Stashko, R. Poberezhnyuk V, V. Vovchenko, M. Gorenstein, I, H. Stoecker, Bose–Einstein condensation phenomenology in systems with repulsive interactions, Phys. Rev. C 102 (3) (2020) http://dx.doi.org/10.1103/PhysRevC.102.035202.
[18] L. Salasnich, BEC in nonestensive statistical mechanics, Internat. J. Modern Phys. B 14 (04) (2000) 405–409, http://dx.doi.org/10.1142/S0217979200000388.
[19] L. Salasnich, BEC in nonestensive statistical mechanics: some additional results, Internat. J. Modern Phys. B 15 (09) (2001) 1253–1256, http://dx.doi.org/10.1142/S0217979201004708.
[20] J. Chen, Z. Zhang, G. Su, L. Chen, Y. Shu, q-generalized Bose–Einstein condensation based on tsallis entropy, Phys. Lett. A 300 (1) (2002) 65–70, http://dx.doi.org/10.1016/S0375-9601(02)00781-8.
[21] O. Penrose, L. Onsager, Bose–Einstein condensation and liquid helium, Phys. Rev. 104 (3) (1956) 576–584, http://dx.doi.org/10.1103/PhysRev.104.576.
[22] F. Abdullaev, A. Gammal, L. Tomio, T. Frederico, Stability of trapped Bose–Einstein condensates, Phys. Rev. A 63 (4) (2001) 66, http://dx.doi.org/10.1103/PhysRevA.63.043604.
[23] W. Ketterle, N. vanDruten, Bose–Einstein condensation of a finite number of particles trapped in one or three dimensions, Phys. Rev. A 54 (1) (1996) 656–660, http://dx.doi.org/10.1103/PhysRevA.54.656.
[24] J.A.S. Lima, A. Deppman, Tsallis meets Boltzmann: q-index for a finite ideal gas and its thermodynamic limit, Phys. Rev. E 101 (4) (2020) http://dx.doi.org/10.1103/PhysRevE.101.040102.
[25] T.S. Biro, G.G. Barnafoeldi, P. Van, Quark-gluon plasma connected to finite heat bath, Eur. Phys. J. A 49 (9) (2013) http://dx.doi.org/10.1140/epja/i2013-13110-0.
[26] G. Alexander, Bose–Einstein and Fermi-Dirac interferometry in particle physics, Rep. Progr. Phys. 66 (4) (2003) 481–522, http://dx.doi.org/10.1088/0034-4885/66/4/202.
[27] D. Kharzeev, E. Levin, K. Tuchin, Multiparticle production and thermalization in high-energy QCD, Phys. Rev. C 75 (4) (2007) http://dx.doi.org/10.1103/PhysRevC.75.044903.
[28] M.G. Alford, A. Schmitt, K. Rajagopal, T. Schaefer, Color superconductivity in dense quark matter, Rev. Modern Phys. 80 (4) (2008) 1455–1515, http://dx.doi.org/10.1103/RevModPhys.80.1455.
[29] H. Nematollahi, K. Javidan, M.M. Yazdanpanah, The QCD strong coupling constant at low energies: a non-extensive treatment, Eur. Phys. J. A 57 (2) (2021) http://dx.doi.org/10.1140/epja/s10050-021-00391-1.
[30] E.P. Borges, A possible deformed algebra and calculus inspired in nonextensive thermostatistics, Physica A 340 (1) (2004) 95–101, http://dx.doi.org/10.1016/j.physa.2004.03.082, news and Expectations in Thermostatistics.
[31] A. Deppman, Thermofractals, non-additive entropy, and q-calculus, Physics 3 (2) (2021) 290–301, http://dx.doi.org/10.3390/physics3020021.
[32] J. Rozynek, G. Wilk, Nonextensive quasiparticle description of QCD matter, Symmetry-Basel 11 (3) (2019) http://dx.doi.org/10.3390/sym11030401.
[33] M.A. Kristensen, M.B. Christensen, M. Gajdacz, M. Iglicki, K. Pawlowski, C. Klempt, J.F. Sherson, K. Rzazewski, A.J. Hilliard, J.J. Arlt, Observation of atom number fluctuations in a Bose–Einstein condensate, Phys. Rev. Lett. 122 (16) (2019) http://dx.doi.org/10.1103/PhysRevLett.122.163601.
[34] M. Huang, A. Bonasera, S. Zhang, H. Zheng, D.X. Wang, J.C. Wang, N. Song, X. Tang, L. Lu, G. Zhang, Z. Kohley, M.R.D. Rodrigues, Y.G. Ma, S.J. Yennello, Four alpha correlations in nuclear fragmentation: a game of resonances, Chin. Phys. C 45 (2) (2021) http://dx.doi.org/10.1088/1674-1137/abce52.
[35] I. Bautista, C. Pajares, J.E. Ramirez, String percolation in AA and p plus p collisions, Rev. Mexicana Fís. 65 (3) (2019) 197–223, http://dx.doi.org/10.31349/RevMexFis.65.197.
[36] S. Deb, D. Sahu, S. Raghunath, A.K. Pradhan, Bose-Einstein condensation of pions in proton–proton collisions at the large hadron collider using non-extensive tsallis statistics, Eur. Phys. J. A 57 (2021) http://dx.doi.org/10.1140/epja/s10050-021-00464-1.
[37] M. Ruggieri, F. Scardina, S. Plumari, V. Greco, Thermalization, isotropization, and elliptic flow from nonequilibrium initial conditions with a saturation scale, Phys. Rev. C 89 (5) (2014) http://dx.doi.org/10.1103/PhysRevC.89.054914.
[38] E. Andrade II, A. Deppman, E. Megias, D.P. Menezes, T.N. da Silva, Bag-type model with fractal structure, Phys. Rev. D 101 (5) (2020) http://dx.doi.org/10.1103/PhysRevD.101.054022.
[39] T. Malik, S. Banik, D. Bandyopadhyay, Equation-of-state table with hyperon and antikaon for supernova and neutron star merger, Agron. J. 910 (2) (2021) http://dx.doi.org/10.3847/1538-4357/abe860.
[40] P.–H. Chavanis, Collapse of a self-gravitating Bose–Einstein condensate with attractive self-interaction, Phys. Rev. D 94 (8) (2016) http://dx.doi.org/10.1103/PhysRevD.94.083007.
[41] E. Gotsman, E. Levin, U. Maor, CGC/saturation Approach for high energy soft interactions: 'soft' pomeron structure and v(n) in hadron and nucleus collisions from Bose–Einstein correlations, Eur. Phys. J. C 76 (11) (2016) http://dx.doi.org/10.1140/epjc/s10052-016-4434-9.
[42] D.P. Menezes, A. Deppman, E. Megias, L.B. Castro, Non-extensive thermodynamics and neutron star properties, Eur. Phys. J. A 51 (12) (2015) http://dx.doi.org/10.1140/epja/i2015-15155-3.
[43] Y. Tang, S. Bao, W. Guo, Superdiffusion of quantized vortices uncovering scaling laws in quantum turbulence, Proc. Natl. Acad. Sci. USA 118 (6) (2021) http://dx.doi.org/10.1073/pnas.2021957118.
[44] V. Filho, A. Gammal, T. Frederico, L. Tomio, Chaos in collapsing bose-condensed gas, Phys. Rev. A 62 (3) (2000) http://dx.doi.org/10.1103/PhysRevA.62.033605.
[45] R. Roy, A. Gammal, M.C. Tsatsos, B. Chatterjee, B. Chakrabarti, A.U.J. Lode, Phases, many-body.entropy. measures, Many-body entropy measures and coherence of interacting bosons in optical lattices, Phys. Rev. A 97 (4) (2018) http://dx.doi.org/10.1103/PhysRevA.97.043625.
[46] T. Nikuni, M. Oshikawa, A. Oosawa, H. Tanaka, Bose-Einstein condensation of dilute magnons in TlCuCl$_3$, Phys. Rev. Lett. 84 (2000) 5868–5871, http://dx.doi.org/10.1103/PhysRevLett.84.5868.
[47] E.Y. Sherman, P. Lemmens, B. Busse, A. Oosawa, H. Tanaka, Sound attenuation study on the bose-einstein condensation of magnons in TlCuCl$_3$, Phys. Rev. Lett. 91 (2003) 057201, http://dx.doi.org/10.1103/PhysRevLett.91.057201.
[48] G. Misguich, M. Oshikawa, Bose-Einstein condensation of magnons in TlCuCl3: Phase diagram and specific heat from a self-consistent hartree–fock calculation with a realistic dispersion relation, J. Phys. Soc. Japan 73 (12) (2004) 3429–3434, http://dx.doi.org/10.1143/JPSJ.73.3429.